Angular momentum dynamics of a paraxial beam in a uniaxial crystal

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The conservation law governing the dynamics of the radiation angular momentum component along the optical axis (z axis) of a uniaxial crystal is derived from Maxwell's equations; the existence of this law is physically related to the rotational invariance of the crystal around the optical axis. Specializing the obtained general expression for the z component of the angular momentum flux to the case of a paraxial beam propagating along the optical axis, we find that the expression is the same as the corresponding one for a paraxial beam propagating in an isotropic medium of refractive index n_o (ordinary refractive index of the crystal); besides, we show that the flux is conserved during propagation and that it decomposes into the sum of an intrinsic and an orbital contribution. Investigating their dynamics we demonstrate that they are coupled and, during propagation, an exchange between them exists. This exchange asymptotically exhibits a saturation process leading, for $z \to \infty$, the intrinsic part to vanish and the orbital one equates the total amount of angular momentum flux. As an example, the evolution of the intrinsic and the orbital contributions to the flux is investigated in the case of circularly polarized beams. Besides, the radiation angular momentum stored in the crystal is also investigated, in the paraxial regime, showing that it is simply given by the product of the total angular momentum flux by the time the radiation takes in passing through the crystal.

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I. INTRODUCTION

Among the physical properties of the electromagnetic field, probably the most relevant ones are that it carries energy, linear momentum, and angular momentum; generally speaking, these are the very features allowing us to regard the electromagnetic field as a physical reality, and not as a merely mathematical machinery set up to describe the interaction among charges. The investigation of the electromagnetic energy and momentum has played an important role since the early development of the electromagnetic theory [1–3]; besides, quantum mechanics has furnished a more exciting picture, associating at each photon an energy $\hbar\omega$, a linear momentum $\hbar\mathbf{k}$, and an angular momentum $\pm\hbar$ (depending on its state of polarization) [4].

The investigation of energy and momentum is simple for an electromagnetic field propagating in vacuum. On the contrary, the problem of deriving the conservation laws for energy, linear and angular momentum associated with a field in a material medium is a subtle one and it has been largely investigated [5]. If we consider the isolated system composed of matter and radiation, it is evident that the total energy, momentum, and angular momentum are conserved quantities, because of the homogeneity of the time, the homogeneity and isotropy of the space, respectively. In the frame of electrodynamics of continuous media, the difficulties arise when we try to express each conserved quantity as the sum of two contributions, one due to the matter and another due to the radiation, resulting in a substantial ambiguity in the definition of each quantity. As an example, following the standard Minkowski approach, one obtains an expression for the Maxwell stress tensor which, for anisotropic media, is not symmetric; this generates serious difficulties about the definition of the angular momentum density and flux. A way of escaping from these shortcomings consists in resorting to a more refined treatment of the interaction between matter and radiation based on a statistical-mechanical approach [6].

Notwithstanding the difficulty of defining what is meant with angular momentum of the light propagating in an anisotropic medium, the investigation of its behavior is expected to be very interesting. In fact the angular momentum of radiation generally has an intrinsic part (or spin), associated with the polarization, and an orbital one associated with the spatial distribution [7]. When propagating in vacuum (or in an isotropic medium) the light does not suffer a change in its polarization state, so that its intrinsic angular momentum is constant; therefore, in spite of diffraction, the orbital part is also constant because the total angular momentum is a conserved quantity. On the other hand, one of the peculiarities of anisotropic media is the change of the state of polarization of

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the light, so that we expect the radiation intrinsic angular momentum to undergo an evolution. Besides, we expect also a nontrivial dynamics in the orbital part, because of the coupling between polarization dynamics and diffraction in anisotropic media.

This behavior strongly encourages us to investigate the angular momentum of radiation in anisotropic media. Nevertheless, if one does not want to overcomplicate the problem (for example, resorting to statistical mechanics), it is essential to consider a medium exhibiting a rotational invariance around some directions. In fact, it is well known that the conservation of the component of the angular momentum along a direction is intimately related to the rotational invariance of the system around the same direction. This implies that, if the medium is rotationally invariant around the z axis, the z component of its angular momentum is conserved so that it cannot be coupled with the z component of the light angular momentum; this absence of coupling establishes a clear distinction between the matter and radiation contributions, so that we expect to be able to precisely define the z component of the radiation angular momentum. The only anisotropic media possessing a rotational symmetry are uniaxial crystals and the rotational invariance is around the optical axis, so that we are led to investigate the dynamics of the component of the light angular momentum along the optical axis.

In the present paper we derive, from macroscopic Maxwell's equations, a balance equation governing the conservation of the z component of the angular momentum of radiation propagating in a uniaxial crystal, whose optical axis coincides with the z axis. The angular momentum density coincides with the classical one derived by Minkowski, while we furnish a different expression for the flux of angular momentum. The treatment is fully electromagnetic with no approximations. Specializing to a situation closer to optics, we consider the case of paraxial light beams propagating along the optical axis, whose behavior has been recently studied [8]. The investigation of the angular momentum of paraxial beams propagating in vacuum is a subject which has attracted much research interest recently. In fact, it has been demonstrated by Allen *et al.* [9,10] that paraxial Laguerre-Gaussian beams [11] carry a well-defined angular momentum. The expressions for the intrinsic and orbital parts of the angular momentum of a paraxial beam in vacuum have been derived by van Enk and Nienhuis [12].

Particularizing our general expression for the angular momentum flux to the case of a paraxial beam, we find that the flux is the same at every transverse section of the beam; besides, we show that the expressions for the intrinsic and orbital parts are the same as those pertinent to a beam propagating in an isotropic medium of refractive index n_o (the ordinary refractive index). This correspondence is physically related to the very structure of the paraxial field and to the rotational invariance of the medium around the optical axis. In spite of this formal similarity between the isotropic and the anisotropic case, we show that the evolution in the crystal of the intrinsic and the orbital part of the angular momentum is fundamentally different from the corresponding isotropic one. In particular we show that the intrinsic and orbital contributions change during the beam propagation, their sum remaining constant. This implies that the polarization-diffraction dynamics of the beam in the crystal yields to an exchange of angular momentum between the intrinsic and orbital contributions. Investigating the asymptotics of this exchange, we show that, for any beam, the intrinsic part vanishes at infinity while the orbital part equates the total angular momentum flux.

In order to test our predictions, we consider the case of circularly polarized beams, recently investigated [13]. This class of beams is particularly suitable for our purposes as we demonstrate that their intrinsic part of the angular momentum flux is simply proportional to the difference between the energies of the left-hand and the right-hand circular component. Besides, in the case of an input left-hand circularly polarized Gaussian beam, we analytically find that the saturation of the exchange between the intrinsic and the orbital part exhibits a Lorentzian profile.

The beam angular momentum stored in the volume of the crystal is also considered and we find that it amounts to the total angular momentum flown through the entrance facet of the crystal in a time equal to that an ordinary plane wave takes to cover the crystal length.

II. RADIATION ANGULAR MOMENTUM ALONG THE OPTICAL AXIS

Let us consider an arbitrary monochromatic electromagnetic field propagating in a uniaxial crystal,

$$\mathbf{E}(\mathbf{r},t)=\mathrm{Re}[\mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}],$$

$$\mathbf{D}(\mathbf{r},t)=\varepsilon_0\varepsilon_r\mathbf{E}(\mathbf{r},t),$$

$$\mathbf{B}(\mathbf{r},t)=\mathrm{Re}[\mathbf{B}_\omega(\mathbf{r})e^{-i\omega t}],$$

$$\mathbf{H}(\mathbf{r},t)=\frac{1}{\mu_0}\mathbf{B}(\mathbf{r},t), \tag{1}$$

where $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$ are the electric, the electric displacement, the magnetic induction, and the magnetic field, respectively, $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$ are the electric and magnetic complex amplitudes, ε_0 and μ_0 are the dielectric and permittivity vacuum constants, respectively, while ε_r is the relative dielectric tensor given by

$$\varepsilon_r=\begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}, \tag{2}$$

n_o and n_e being the ordinary and extraordinary refractive indices respectively, for the frequency ω; note that the reference frame has been chosen so that the optical axis of the crystal coincides with the z axis. In the present approach we neglect the absorbtion, so that n_o and n_e are real numbers, allowing the second of Eqs. (1) to hold.

The first step of our analysis consists in obtaining, from Maxwell's equations, the proper expression for the component along the optical axis of the angular momentum of the field and its flux. Let us start by considering an arbitrary volume τ, inside the crystal, filled by charges, described by a volume density ρ and current density $\mathbf{J}$. Because of the interaction with the electromagnetic field, the charges experience a total mechanical torque $\mathbf{M}$ given by [14]

$$\mathbf{M}=\int_{\tau} d\mathbf{r}\, \mathbf{r}\times(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}). \tag{3}$$

Following the standard procedure, we manipulate Eq. (3) by employing Maxwell's equations

$$\begin{aligned}\nabla\cdot\mathbf{D}&=\rho,\\ \nabla\cdot\mathbf{B}&=0,\\ \nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t},\\ \nabla\times\mathbf{H}&=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t};\end{aligned} \tag{4}$$

substituting ρ and $\mathbf{J}$ from the inhomogeneous equations [the first and the fourth of Eqs. (4)] into Eq. (3) and rearranging the obtained expression by means of the homogenous equations [the second and the third of Eqs. (4)], we get

$$\mathbf{M}+\frac{d}{dt}\int_{\tau} d\mathbf{r}\,\mathbf{r}\times(\mathbf{D}\times\mathbf{B})=\int_{\tau} d\mathbf{r}\,\mathbf{r}\times(\mathbf{E}\nabla\cdot\mathbf{D}-\mathbf{D}\times\nabla\times\mathbf{E})+\int_{\tau} d\mathbf{r}\,\mathbf{r}\times(\mathbf{H}\nabla\cdot\mathbf{B}-\mathbf{B}\times\nabla\times\mathbf{H}). \tag{5}$$

Since the mechanical torque $\mathbf{M}$ is equal to the rate of variation of the angular momentum $\mathbf{L}_c$ of the charges ($\mathbf{M}=d\mathbf{L}_c/dt$), we are tempted to read Eq. (5) as a balance equation for the total angular momentum and to interpret the quantity

$$\mathbf{L}_f=\int_{\tau} d\mathbf{r}\,\mathbf{r}\times(\mathbf{D}\times\mathbf{B}) \tag{6}$$

as the angular momentum of the electromagnetic field stored in the volume τ. However, this is possible *if* and *only if* the right-hand side (RHS) of Eq. (5) can be transformed into a surface integral over the boundary $\Sigma=\partial\tau$ enclosing the volume τ; only in this case Eq. (5) would equate the rate of increasing of the total angular momentum $\mathbf{L}=\mathbf{L}_c+\mathbf{L}_f$ stored in the volume τ to the flux of incoming angular momentum through the surface Σ. It is a well-known fact that for linear isotropic media, the RHS of Eq. (5) can be expressed as a surface integral, but, for the case of linear anisotropic media we are investigating, the situation is more involved.

Taking Eqs. (1) into account and exploiting the symmetry of the tensor ε_r, it is straightforward to prove that

$$\mathbf{E}\nabla\cdot\mathbf{D}-\mathbf{D}\times\nabla\times\mathbf{E}+\mathbf{H}\nabla\cdot\mathbf{B}-\mathbf{B}\times\nabla\times\mathbf{H}=\nabla\cdot T, \tag{7}$$

where T is the well-known Maxwell stress tensor, given by

$$T_{ij}=\epsilon_0 E_i(\varepsilon_r\mathbf{E})_j+\frac{1}{\mu_0}B_iB_j-\frac{1}{2}\left(\varepsilon_0\mathbf{E}\cdot\varepsilon_r\mathbf{E}+\frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right)\delta_{ij}, \tag{8}$$

the indices i and j running over the number $1=x$, $2=y$, and $3=z$ and δ_{ij} being the Kronecker delta function; the divergence of the tensor is defined as a vector whose components are given by $(\nabla\cdot T)_i=\partial_j T_{ij}$ where the convention of summing over the repeated indices has been used. Note that

$$T-T^t=\epsilon_0(n_o^2-n_e^2)E_z\begin{pmatrix}0 & 0 & -E_x\\ 0 & 0 & -E_y\\ E_x & E_y & 0\end{pmatrix}, \tag{9}$$

where the superscript t indicates the transposition operation. Equation (9) shows that the tensor T is not symmetric because $n_o\neq n_e$, that is to say because of the anisotropy of the medium. The RHS of Eq. (5) is the volume integral of the vector $\mathbf{r}\times\nabla\cdot T$ which can be rewritten as

$$\mathbf{r}\times\nabla\cdot T=\nabla\cdot F+\mathbf{g}, \tag{10}$$

where the tensor F and the vector $\mathbf{g}$ are defined by

$$\begin{aligned}F_{ij}&=\epsilon_{imn}x_m T_{nj},\\ g_i&=\epsilon_{imn}T_{mn},\end{aligned} \tag{11}$$

and ϵ_{imn} is the completely antisymmetric tensor of rank 3 (Levi-Civitá symbol). Inserting Eqs. (6) and (10) into Eq. (5) and exploiting the Green's theorem to transform the volume integral into a surface one, we obtain

$$\frac{d}{dt}(\mathbf{L}_c+\mathbf{L}_f)=\int_{\Sigma} dS\, F\hat{\mathbf{n}}+\int_{\tau} d\mathbf{r}\,\mathbf{g}, \tag{12}$$

where $\hat{\mathbf{n}}=n_x\hat{\mathbf{e}}_x+n_y\hat{\mathbf{e}}_y+n_z\hat{\mathbf{e}}_z$ is the unit vector pointing outward from the surface. Equation (12) can be interpreted as the balance equation for the angular momentum only if the volume integral of the vector $\mathbf{g}$ vanishes. In this perspective, it is crucial to inspect the explicit expression of $\mathbf{g}$ which, from the second of Eqs. (11) and Eq. (8), is given by

$$\mathbf{g}=\begin{pmatrix}T_{yz}-T_{zy}\\ -T_{xz}+T_{zx}\\ T_{xy}-T_{yx}\end{pmatrix}=\epsilon_0(n_o^2-n_e^2)E_z\begin{pmatrix}-E_y\\ E_x\\ 0\end{pmatrix}. \tag{13}$$

This expression shows that $\mathbf{g}$, in general, does not vanish and this is a consequence of the fact that the stress tensor T is not symmetric or, equivalently, that the medium we are considering is anisotropic. Therefore Eq. (12) does not allow us to interpret $\mathbf{L}_f$ and the surface integral as the angular momentum and the flux of angular momentum, respectively, since there is an additional uninterpreted contribution in the balance equation. However, the z component of $\mathbf{g}$ vanishes because $T_{xy}=T_{yx}$ as a consequence of the rotational invariance around the z axis of the crystal. Therefore, if we consider only the z component of Eq. (12), we obtain

$$\frac{d}{dt}(L_{cz}+L_{fz})=\int_{\Sigma}dS(F_{zx}n_x+F_{zy}n_y+F_{zz}n_z), \quad (14)$$

which is an expression exhibiting the desired structure; therefore we are allowed to interpret the scalar L_{fz} as the z component of the angular momentum of the radiation stored in the volume τ and the surface integral in the RHS of Eq. (14) as the flux of angular momentum incoming through the boundary Σ.

The obtained result deserves some discussion. At a first glance, it can appear strange that we were easily able to tackle the z component of the angular momentum and, on the contrary, not able to give a complete description of the x and the y components. The origin of this difference is intimately related to the rotational invariance around the z axis of the uniaxial crystal. Propagating through the medium, the electromagnetic field exchanges angular momentum not only with the charges ρ but also with the crystal. The charges contribution to the angular momentum is $\mathbf{L}_c$ by definition, but it is not simple to properly distinguish the contribution of the crystal from that one of the electromagnetic field, a task requiring more refined methods based, for example, on a statistical-mechanical approach to the interaction between matter and radiation. The consequence of this difficulty is the appearance of the volume integral of the vector $\mathbf{g}$ in the RHS of Eq. (12). Therefore, in general, we are not able to give a correct definition of the angular momentum of the radiation only within the frame of electrodynamics of macroscopic media. However, the situation we are analyzing is particular in the sense that the crystal we are considering shows a rotational symmetry around the optical axis. It is well known that the existence of a rotational symmetry around a direction is related to the conservation of the projection of the angular momentum of the system along the same direction. This implies that the angular momentum of the crystal along z is a conserved quantity not mixing with the angular momentum of the light; the rotational invariance around the z axis decouples the contribution of the crystal and that of the field, allowing us to correctly distinguish them.

III. ANGULAR MOMENTUM FLUX OF A PARAXIAL BEAM PROPAGATING ALONG THE OPTICAL AXIS

The situation investigated in the above section is valid for any monochromatic electromagnetic field propagating in a uniaxial crystal. In this section, we want to apply the general treatment to the particular case of a paraxial beam propagating along the optical axis of the crystal; let us now specialize Eq. (14) to this case. The standard situation in optics is that of light traveling in a medium without charges; this implies that we can set $L_{cz}=0$. Besides, since the beam propagates along the z axis, we are interested in the total amount of the z component of the angular momentum of beam flowing through any plane $z=z_0$; therefore we choose the volume τ to be the stripe $0<z<z_0$. With these prescriptions, Eq. (14) becomes

$$\frac{dL_{fz}}{dt}=-\int d^2\mathbf{r}_\perp F_{zz}(\mathbf{r}_\perp,0,t)+\int d^2\mathbf{r}_\perp F_{zz}(\mathbf{r}_\perp,z_0,t), \quad (15)$$

where $\mathbf{r}_\perp=x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y$, $d^2\mathbf{r}_\perp=dxdy$, and the integrals are extended over the whole transverse planes; note that we have neglected the contributions to the integral coming from the surface at infinity since the paraxial beam is always a transverse localized entity and it rapidly vanishes for $|\mathbf{r}_\perp|\to\infty$. The field we are investigating is monochromatic so that, rather than considering the instantaneous value of the flux of angular momentum it is more convenient to investigate the correspondent time average. In this perspective, we take the time average of Eq. (15), that is to say,

$$\left\langle\frac{dL_{fz}}{dt}\right\rangle=-\int d^2\mathbf{r}_\perp\langle F_{zz}(\mathbf{r}_\perp,0,t)\rangle+\int d^2\mathbf{r}_\perp\langle F_{zz}(\mathbf{r}_\perp,z_0,t)\rangle, \quad (16)$$

where the time average $\langle f\rangle$ of any quantity $f(t)$ is defined as

$$\langle f\rangle=\frac{1}{T_I}\int_{-T_I/2}^{T_I/2}dt f(t) \quad (17)$$

and T_I is an integration time much greater than the period $2\pi/\omega$ of the radiation. Since the time average of a time derivative vanishes, we get, from Eq. (16) and from the arbitrariness of z_0, the relation

$$\frac{d\Phi(z)}{dz}=0, \quad (18)$$

where we defined

$$\Phi(z)=-\int d^2\mathbf{r}_\perp\langle F_{zz}(\mathbf{r}_\perp,z,t)\rangle. \quad (19)$$

$\Phi(z)$ has the physical meaning of the time-averaged flux of the z component of the angular momentum of the beam flowing through any fixed z plane from left to right. Equation (18) expresses the conservation of the z component of the angular momentum. Note that the presence of the minus sign in the definition of $\Phi(z)$ is conventional and it is due to the fact that the integral of $\langle F_{zz}\rangle$ represents a flux of angular momentum flowing from right to left, while we find more familiar to think that the beam propagates from left to right.

To go further, we have now to compute the time average $\langle F_{zz}\rangle$,

$$\langle F_{zz}\rangle=\langle -yT_{xz}+xT_{yz}\rangle=-y\left(\varepsilon_0 n_e^2\langle E_xE_z\rangle+\frac{1}{\mu_0}\langle B_xB_z\rangle\right)+x\left(\varepsilon_0 n_e^2\langle E_yE_z\rangle+\frac{1}{\mu_0}\langle B_yB_z\rangle\right), \quad (20)$$

where use has been made of the first of Eqs. (11) and of Eq. (8). The time average of the product of two monochromatic fields is easily given by the half of the real part of the product between the complex amplitude of one of them and the conjugate of the complex amplitude of the other; therefore Eq. (20) becomes

$$\langle F_{zz}\rangle = \frac{1}{2}\mathrm{Re}[\varepsilon_0 n_e^2(-yE_{\omega x}+xE_{\omega y})E_{\omega z}^*] + \frac{1}{2}\mathrm{Re}\left[\frac{1}{\mu_0}(-yB_{\omega x}+xB_{\omega y})B_{\omega z}^*\right], \tag{21}$$

where $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$ are the electric and magnetic complex amplitudes defined in Eqs. (1). Using Maxwell's equations of Eqs. (4) (where we set $\rho=0$ and $\mathbf{J}=\mathbf{0}$) and Eqs. (1), it is easy to prove that

$$B_{\omega z} = \frac{1}{i\omega}\left(\frac{\partial E_{\omega y}}{\partial x} - \frac{\partial E_{\omega x}}{\partial y}\right),$$

$$E_{\omega z} = \frac{i}{\omega \varepsilon_0 \mu_0 n_e^2}\left(\frac{\partial B_{\omega y}}{\partial x} - \frac{\partial B_{\omega x}}{\partial y}\right). \tag{22}$$

Inserting Eqs. (22) into Eq. (21) we succeed in eliminating the longitudinal components of the electromagnetic field, thus obtaining

$$\langle F_{zz}\rangle = \frac{1}{2\omega\mu_0}\mathrm{Re}\left[\frac{1}{i}(-yE_{\omega x}+xE_{\omega y})\left(\frac{\partial B_{\omega y}^*}{\partial x} - \frac{\partial B_{\omega x}^*}{\partial y}\right)\right] + \frac{1}{2\omega\mu_0}\mathrm{Re}\left[i(-yB_{\omega x}+xB_{\omega y})\left(\frac{\partial E_{\omega y}^*}{\partial x} - \frac{\partial E_{\omega x}^*}{\partial y}\right)\right]. \tag{23}$$

Up to now, all the expressions we have considered are exact and valid for any monochromatic field. Let us now restrict our attention to the set of paraxial beams (propagating along the z axis), for which the transverse size is much greater than the wavelength. These kind of fields have been extensively investigated [8,15–18] and the complex amplitude of the electric field can be expressed as [8]

$$\mathbf{E}_\omega(\mathbf{r}_\perp, z) \simeq e^{ik_0 n_o z}\mathbf{A}_\perp(\mathbf{r}_\perp, z), \tag{24}$$

where $k_0=\omega/c$ and the field $\mathbf{A}_\perp = A_x\hat{\mathbf{e}}_x + A_y\hat{\mathbf{e}}_y$ is a slowly varying transverse amplitude. From the third Maxwell's equation we obtain the transverse components of the magnetic induction field, that is to say

$$B_{\omega x}(\mathbf{r}_\perp, z) \simeq -\frac{k_0 n_o}{\omega}e^{ik_0 n_o z}A_y(\mathbf{r}_\perp, z),$$

$$B_{\omega y}(\mathbf{r}_\perp, z) \simeq \frac{k_0 n_o}{\omega}e^{ik_0 n_o z}A_x(\mathbf{r}_\perp, z), \tag{25}$$

where we have neglected the transverse derivative of $E_{\omega z}$ since the longitudinal component is much smaller than the transverse part $\mathbf{E}_{\omega\perp}$, in paraxial regime. Substituting Eqs. (24) and (25) into Eq. (23), we get

$$\langle F_{zz}\rangle = \frac{k_0 n_o}{2\omega^2\mu_0}\mathrm{Re}\left[\frac{1}{i}(-yA_x+xA_y)\left(\frac{\partial A_x^*}{\partial x} + \frac{\partial A_y^*}{\partial y}\right) + i(yA_y+xA_x)\left(\frac{\partial A_y^*}{\partial x} - \frac{\partial A_x^*}{\partial y}\right)\right]. \tag{26}$$

Note that this expression relates, the average density current of z component of the angular momentum of the beam only to its slowly varying amplitude $\mathbf{A}_\perp$. Substituting Eq. (26) into Eq. (19) we obtain, after some manipulations (see Sec. 1 of the Appendix),

$$\Phi(z) = n_o\frac{\varepsilon_0 c}{2\omega}\left[i\int d^2\mathbf{r}_\perp(A_xA_y^* - A_x^*A_y) + \frac{1}{i}\int d^2\mathbf{r}_\perp A_x^*\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)A_x + \frac{1}{i}\int d^2\mathbf{r}_\perp A_y^* \times\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)A_y\right]. \tag{27}$$

This equation constitutes one of the main results of the present paper. It is interesting to note that this expression coincides with that one valid for a paraxial beam propagating in a homogeneous isotropic medium with refractive index n_o [7,12]. The physical interpretation of this intriguing coincidence is related to the very structure of a paraxial field and, again, to the rotational invariance around the z axis. A paraxial beam propagating in an isotropic medium can be thought of as a superposition of plane waves whose wave vectors slightly differ from the main one that gives the main direction of propagation, say the z axis. This implies that the beam can be expressed as $\mathbf{E}_\omega = \exp(ik_0nz)\mathbf{A}_\perp$ where n is the refractive index and $\mathbf{A}_\perp$ is a slowly varying amplitude; the field is then a plane wave carrier modulated by an envelope. A paraxial beam propagating in a uniaxial crystal is also a superposition of plane waves whose wave vectors are nearly parallel to a main direction, say $\hat{\mathbf{s}}$, but, due to the existence of two different kinds of plane waves (the ordinary and extraordinary waves), in a given direction there are two main wave vectors. This implies that the field can be expressed as $\mathbf{E}_\omega = \exp[ik_0n_o(\hat{\mathbf{s}})z]\mathbf{A}_{\perp o} + \exp[ik_0n_e(\hat{\mathbf{s}})z]\mathbf{A}_{\perp e}$, where $n_o(\hat{\mathbf{s}})$ and $n_e(\hat{\mathbf{s}})$ are the ordinary and extraordinary refractive indices for the direction $\hat{\mathbf{s}}$. Therefore, the beam in the isotropic medium and that in the uniaxial crystal show, in general, different structures and consequently no simple relation can exist between their angular momenta. However, for a beam propagating along the optical axis, the situation is very particular since in this case $\hat{\mathbf{s}} = \hat{\mathbf{e}}_z$ and, due to the rotational invariance around the z axis we have $n_o(\hat{\mathbf{e}}_z) = n_e(\hat{\mathbf{e}}_z) = n_o$, that is to say the two carriers share a common wave vector, experiencing the same refractive index n_o; this allows us to express the field as in Eq. (24). Therefore, the beam along the optical axis exhibits the same structure of a beam propagating in an isotropic medium of refractive index n_o. The coincidence of the expressions for the flux of the z component of the angular momentum in an isotropic medium and in the crystal follows immediately by observing that Eq. (23) (which is exact) does not depend on the refractive indices and consequently it holds for both an isotropic medium and the uniaxial crystal.

Another interesting property of Eq. (27), which has been pointed out in the case of isotropic media [7,12], is retrieved by noting that the flux Φ can be expressed as

$$\Phi(z)=\Phi_I(z)+\Phi_O(z), \tag{28}$$

having set

$$\Phi_I(z)=i\gamma\int d^2\mathbf{r}_\perp \mathbf{A}_\perp^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{A}_\perp ,$$

$$\Phi_O(z)=\frac{\gamma}{i}\int d^2\mathbf{r}_\perp \mathbf{A}_\perp^\dagger\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\mathbf{A}_\perp , \tag{29}$$

where the superscript $\dagger$ indicates the Hermitian conjugation operation and we defined for convenience

$$\gamma=n_o\frac{\varepsilon_0 c}{2\omega}. \tag{30}$$

From Eq. (28) we observe that the flux of the angular momentum is the sum of two contributions, Φ_I and Φ_O which are commonly called the intrinsic (or spin) and the orbital part, respectively, of the angular momentum. The first one is mainly related to the state of polarization of the field whereas the second is essentially related to the shape of the field.

IV. EVOLUTION OF THE INTRINSIC AND ORBITAL ANGULAR MOMENTUM FLUXES

In the above section, we have demonstrated that the expressions for the angular momentum flux Φ of a paraxial beam propagating in a uniaxial crystal (along the optical axis) and in an isotropic medium are the same. The main common features are that the flux is conserved [see Eq. (18)] and that it is given by the sum of the intrinsic and the orbital contributions, Φ_I and Φ_O, respectively. However, the propagation of a beam in a uniaxial crystal shows some unique features, absent in the isotropic counterpart, the most relevant being the change of the polarization state. We want now to investigate how this effect affects the dynamics of Φ_I and Φ_O.

The expression for a generic paraxial beam propagating along the optical axis of a uniaxial crystal has been deduced in Ref. [8] and it is shown there that its slowly varying amplitude [as defined in Eq. (24)] is given by

$$\mathbf{A}_\perp(\mathbf{r}_\perp,z)=\int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}[e^{-(iz/2k_0n_o)k_\perp^2}\hat{P}_o + e^{-(in_oz/2k_0n_e^2)k_\perp^2}\hat{P}_e]\tilde{\mathbf{A}}_\perp(\mathbf{k}_\perp), \tag{31}$$

where

$$\hat{P}_o=\frac{1}{k_\perp^2}\begin{pmatrix} k_y^2 & -k_xk_y \\ -k_xk_y & k_x^2 \end{pmatrix},\quad \hat{P}_e=\frac{1}{k_\perp^2}\begin{pmatrix} k_x^2 & k_xk_y \\ k_xk_y & k_y^2 \end{pmatrix}, \tag{32}$$

$\mathbf{k}_\perp=k_x\hat{\mathbf{e}}_x+k_y\hat{\mathbf{e}}_y$ and $d^2\mathbf{k}_\perp=dk_xdk_y$. In Eq. (31), the field $\tilde{\mathbf{A}}_\perp$ is related to the boundary distribution of the electric field by means of the relation

$$\tilde{\mathbf{A}}_\perp(\mathbf{k}_\perp)=\frac{1}{(2\pi)^2}\int d^2\mathbf{r}_\perp e^{-i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\mathbf{E}_\perp(\mathbf{r}_\perp,0), \tag{33}$$

which is a standard two-dimensional Fourier transform. In order to study the dynamics of Φ_I and Φ_O, we substitute the field of Eq. (31) into Eqs. (29) and we obtain, after some algebra (see Sec. 2 of the Appendix),

$$\Phi_I(z)=i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp\tilde{\mathbf{A}}_\perp^\dagger[e^{(i\Delta z/2k_0n_o)k_\perp^2}\hat{Q}_o + e^{-(i\Delta z/2k_0n_o)k_\perp^2}\hat{Q}_e]\tilde{\mathbf{A}}_\perp ,$$

$$\begin{aligned}\Phi_O(z)=&-i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp\tilde{\mathbf{A}}_\perp^\dagger[e^{(i\Delta z/2k_0n_o)k_\perp^2}\hat{Q}_o + e^{-(i\Delta z/2k_0n_o)k_\perp^2}\hat{Q}_e]\tilde{\mathbf{A}}_\perp \\ &+i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp\tilde{\mathbf{A}}_\perp^\dagger\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\left(k_y\frac{\partial}{\partial k_x}-k_x\frac{\partial}{\partial k_y}\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]\tilde{\mathbf{A}}_\perp ,\end{aligned} \tag{34}$$

where we have set $\Delta=n_o^2/n_e^2-1$ and we have introduced the matrices

$$\hat{Q}_o=\frac{1}{k_\perp^2}\begin{pmatrix} k_xk_y & -k_x^2 \\ k_y^2 & -k_xk_y \end{pmatrix},\quad \hat{Q}_e=\frac{1}{k_\perp^2}\begin{pmatrix} -k_xk_y & -k_y^2 \\ k_x^2 & k_xk_y \end{pmatrix}. \tag{35}$$

Equations (34) constitute the main result of the present paper and they deserve a special discussion. The first striking effect emerging from these equations is that Φ_I and Φ_O depend on z so that they are not conserved quantities; at the same time, their sum $\Phi=\Phi_I+\Phi_O$ does not depend on z, in unavoidable agreement with Eq. (18), expressing the conservation of the total angular momentum flux. Therefore, the dynamics of Φ_I and Φ_O presents an exchange of angular momentum flux between the intrinsic and the orbital contributions. It is worth noting that this exchange is fundamentally mediated by the anisotropy, since in an isotropic medium the intrinsic and the orbital contributions are separately conserved quantities; note that this well-known fact concerning isotropic media can be easily retrieved from Eqs. (34) simply putting $n_o=n_e$ or, equivalently $\Delta=0$.

From a physical point of view, the exchange of angular momentum between the two contributions is easily understood by taking into account the change of the state of polarization of a beam traveling through the crystal. The main consequence of this polarization dynamics on the angular momentum is that the intrinsic part Φ_I generally changes because of its dependence on the polarization state. Since the total angular momentum flux is a conserved quantity, the evolution of Φ_I corresponds to that of Φ_O, explaining the origin of the exchange.

In order to mathematically describe this exchange of angular momentum flux, note that, putting, $z=0$ into Eqs. (34), we get

$$\Phi_I(0)=i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp \tilde{\mathbf{A}}_\perp^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{A}}_\perp ,$$

$$\Phi_O(0)=i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp \tilde{\mathbf{A}}_\perp^\dagger \left(k_y\frac{\partial}{\partial k_x}-k_x\frac{\partial}{\partial k_y}\right)\tilde{\mathbf{A}}_\perp \quad (36)$$

relating the boundary values of Φ_I and Φ_O directly to the spectrum $\tilde{\mathbf{A}}_\perp$. Comparing Eqs. (34) and (36), it is straightforward to see that

$$\Phi_I(z)=\Phi_I(0)-\Delta\Phi(z),$$

$$\Phi_O(z)=\Phi_O(0)+\Delta\Phi(z), \quad (37)$$

where we defined

$$\Delta\Phi(z)=i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp \tilde{\mathbf{A}}_\perp^\dagger \Bigg[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - e^{(i\Delta z/2k_0n_o)k_\perp^2}\hat{Q}_o - e^{-(i\Delta z/2k_0n_o)k_\perp^2}\hat{Q}_e \Bigg]\tilde{\mathbf{A}}_\perp . \quad (38)$$

Equations (37) describe the exchange of angular momentum flux in a particularly transparent way. We recognize in $\Delta\Phi(z)$ the amount of angular momentum flux which, after a distance z, the intrinsic component Φ_I has transferred to the orbital one Φ_O. Note that in the isotropic limit (i.e., $\Delta=0$) $\Delta\Phi$ uniformly vanishes, expressing again the absence of angular momentum flux exchange in isotropic media.

Because of the presence of $\tilde{\mathbf{A}}_\perp$ in the expression for $\Delta\Phi$ of Eq. (38), the evolution of the angular momentum fluxes cannot be characterized in general, since it is strongly dependent on the beam shape. However, there is a particular feature of the angular momenta dynamics which is the same for any beam traveling through the crystal and it is related to the asymptotics of the exchange. In order to discuss this point, note that if z is very large, the integral of Eq. (38) contains two highly oscillatory functions so that their contributions to the integral are expected to be very small; it is possible to prove that for $z\to\infty$ their contributions exactly vanish so that

$$\lim_{z\to\infty}\Delta\Phi(z)=i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp \tilde{\mathbf{A}}_\perp^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\mathbf{A}}_\perp . \quad (39)$$

Note that, because this limit always exists, we find a saturation process in the evolution of the angular momentum fluxes analogous to that one of the energy dynamics of the x and y components of the beam discussed in Ref. [19]. Comparing Eq. (39) with the first of Eqs. (36), we conclude that

$$\lim_{z\to\infty}\Delta\Phi(z)=\Phi_I(0). \quad (40)$$

This is an interesting result since we have demonstrated that the asymptotical amount of the angular momentum flux flowing from the intrinsic contribution to the orbital one coincides with the boundary value of the intrinsic contribution. Computing the limit for $z\to\infty$ in Eqs. (37) and exploiting Eq. (40), we finally obtain

$$\lim_{z\to\infty}\Phi_I(z)=0,$$

$$\lim_{z\to\infty}\Phi_O(z)=\Phi_I(0)+\Phi_O(0)=\Phi, \quad (41)$$

revealing, for these two quantities, a very interesting kind of saturation. Asymptotically, the intrinsic angular momentum flux vanishes whereas the orbital one equates the total amount of angular momentum flux carried by the beam.

V. CIRCULARLY POLARIZED BEAMS

The above considerations are very general and it is difficult to discuss the details of the angular momenta evolution because of the involved dynamics of the polarization state. However, in the particular and interesting case of circularly polarized beams, the analysis can be generally further developed. In order to investigate the angular momentum dynamics of this kind of beams, we introduce the fields

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix}, \quad (42)$$

which are the standard left-hand and right-hand circularly polarized components of the field. The propagation of circularly polarized beams have been recently investigated [13] and it has been shown that they undergo an interesting behavior concerning the coupling among the vortex components of each circular component. Substituting Eq. (42) into Eqs. (29) and using polar coordinates ($x=r\cos\varphi$ and $y=r\sin\varphi$) inside the integral, we obtain

$$\Phi_I(z)=\gamma\int_0^\infty dr\, r\int_0^{2\pi} d\varphi(|A_+|^2-|A_-|^2),$$

$$\Phi_O(z)=\frac{\gamma}{i}\int_0^\infty dr\, r\int_0^{2\pi} d\varphi\left(A_+^*\frac{\partial A_+}{\partial\varphi}+A_-^*\frac{\partial A_-}{\partial\varphi}\right). \quad (43)$$

From the first of Eq. (43), we note that the intrinsic contribution to the angular momentum is proportional to the difference between the energies of left-hand and the right-hand circular components of the beam; the second of Eq. (43) shows that the orbital contribution to the angular momentum is highly related to the dependence of the field components on the polar angle φ. The saturation process of Φ_I outlined in the above section can now be conveniently reinterpreted by means of Eqs. (43). The first of Eqs. (41) and the first of Eqs. (43) allow us to state that the intrinsic contributions to the angular momentum due to A_+ and to A_- are, asymptotically, equal with opposite sign; this is equivalent to saying that the asymptotical values of the energies of A_+ and A_- coincide.

In Ref. [13], the authors considered the particular case of a beam which, on the plane $z=0$, is purely left-hand circularly polarized and exhibits circular symmetry, that is to say

$$E_+(r,\varphi,0)=E(r),$$
$$E_-(r,\varphi,0)=0. \tag{44}$$

It is interesting to investigate the evolution of the angular momenta for this beam since, for $z=0$, its orbital contribution vanishes, $\Phi_O(0)=0$ [see the second of Eqs. (43)], whereas its intrinsic contribution is proportional to the energy of the beam [see the first of Eqs. (43)]. In Ref. [13], it is shown that

$$A_+(r,z)=\pi\int_0^\infty dk\, k[e^{-(iz/2k_0n_o)k^2}+e^{-(in_oz/2k_0n_e^2)k^2}]J_0(kr)\tilde{E}(k),$$
$$A_-(r,\varphi,z)=\pi e^{i2\varphi}\int_0^\infty dk\, k[e^{-(iz/2k_0n_o)k^2}-e^{-(in_oz/2k_0n_e^2)k^2}]J_2(kr)\tilde{E}(k), \tag{45}$$

where J_n is the Bessel function of first kind of order n and

$$\tilde{E}(k)=\frac{1}{2\pi}\int_0^\infty dr\, rJ_0(kr)E(r). \tag{46}$$

Equations (45) show that the left-hand circular component A_+ keeps its circular symmetry in propagation, while the right-hand circular component A_- grows and it carries a topological charge 2, since it depends on φ only by means of the factor $\exp(i2\varphi)$. Substituting Eqs. (45) into Eqs. (43), we obtain

$$\Phi_I(z)=\gamma[W_+(z)-W_-(z)],$$
$$\Phi_O(z)=2\gamma W_-(z), \tag{47}$$

where we have defined the optical powers

$$W_\pm(z)=\int_0^\infty dr\, r\int_0^{2\pi}d\varphi|A_\pm(r,\varphi,z)|^2. \tag{48}$$

From Eqs. (47) we note that the orbital contribution to the angular momentum depends only on the right-hand component and that it grows, while the intrinsic contribution diminishes. Note also that, in this particular case, the total angular momentum

$$\Phi(z)=\Phi_I(z)+\Phi_O(z)=\gamma[W_+(z)+W_-(z)] \tag{49}$$

is proportional to the total energy of the beam which is a constant because the crystal is lossless.

The particular case of a Gaussian beam

$$E(r)=E_0e^{-r^2/2s}, \tag{50}$$

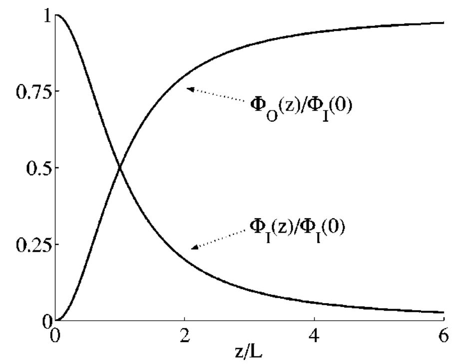

FIG. 1. Plots of $\Phi_I(z)/\Phi_I(0)$ and $\Phi_O(z)/\Phi_I(0)$ of a Gaussian beam vs the normalized propagation distance z/L.

with spot size s, admits an analytical treatment, and we have (see Ref. [13])

$$W_\pm(z)=\frac{1}{2}W_+(0)\left[1\pm\frac{1}{1+\left(\frac{z}{L}\right)^2}\right], \tag{51}$$

where $L=2k_0n_os^2/\Delta$. Substituting Eq. (51) into Eqs. (47) and noting that $\Phi_I(0)=\gamma W_+(0)$, we straightforwardly get

$$\Phi_I(z)=\Phi_I(0)\left[\frac{1}{1+\left(\frac{z}{L}\right)^2}\right],$$
$$\Phi_O(z)=\Phi_I(0)\left[1-\frac{1}{1+\left(\frac{z}{L}\right)^2}\right] \tag{52}$$

exhibiting a Lorentzian saturation; in Fig. 1, the plots of Eqs. (52) are reported.

VI. VOLUME ANGULAR MOMENTUM

For the sake of completeness, we now investigate the amount of z component of the angular momentum of the radiation stored in the crystal that, in the first section of the present paper, we showed to be given by $L_{fz}=\hat{\mathbf{e}}_z\cdot\mathbf{L}_f$. In order to obtain the expression corresponding to a paraxial beam, we consider the z component of Eq. (6),

$$L_{fz}=\varepsilon_0\int_\tau d\mathbf{r}[-n_o^2(xE_x+yE_y)B_z+n_e^2(xB_x+yB_y)E_z)], \tag{53}$$

where the second of Eqs. (1) has been taken into account; in this equation, τ is the volume filled by the crystal. Also in this case, we are interested in the time average of L_{fz} that is given by

$$\langle L_{fz}\rangle=\frac{\varepsilon_0}{2}\mathrm{Re}\int_\tau d\mathbf{r}[-n_o^2(xE_{\omega x}+yE_{\omega y})B^*_{\omega z}+n_e^2(xB_{\omega x}+yB_{\omega y})E^*_{\omega z}], \tag{54}$$

where use has been made of the rule for obtaining the time average of the product of two monochromatic quantities. In order to handle this expression, we follow the same procedure used for the angular momentum flux. Inserting Eqs. (22) into Eq. (54) we eliminate the longitudinal components of the field; subsequently, using Eqs. (24) and (25) for describing a paraxial beam, we obtain

$$\langle L_{fz}\rangle = -\frac{\varepsilon_0 n_o^2}{2\omega}\mathrm{Re}\int_\tau d\mathbf{r}\left[\frac{1}{i}(-yA_x + xA_y)\left(\frac{\partial A_x^*}{\partial x}+\frac{\partial A_y^*}{\partial y}\right) + i(yA_y + xA_x)\left(\frac{\partial A_y^*}{\partial x}-\frac{\partial A_x^*}{\partial y}\right)\right]. \quad (55)$$

Comparing Eq. (55) with Eq. (26), we note that

$$\langle L_{fz}\rangle = -\frac{n_o}{c}\int_\tau d\mathbf{r}\ \langle F_{zz}\rangle, \quad (56)$$

which is a relevant expression relating, in paraxial regime, the radiation angular momentum stored in the crystal to the flux. If we model the crystal as the slab between the plane $z=0$ and $z=D$, D being the length of the crystal, the integrals over $\mathbf{r}_\perp$ and that over z can be splitted so that

$$\langle L_{fz}\rangle = -\frac{n_o}{c}\int_0^D dz\int d^2\mathbf{r}_\perp \langle F_{zz}\rangle = \frac{n_o}{c}\int_0^D dz\Phi(z), \quad (57)$$

where use has been made of Eq. (19) to introduce the flux Φ. Taking Eq. (18) into account (conservation of the total flux Φ), Eq. (57) readily yields

$$\langle L_{fz}\rangle = \frac{n_o D}{c}\Phi, \quad (58)$$

which is the simpler expression we can obtain relating the angular momentum and its flux. Note that $n_o D/c$ is the time that an ordinary plane wave takes to go from $z=0$ to $z=D$; therefore Eq. (58) admits a simple physical interpretation regarding the volume angular momentum $\langle L_{fz}\rangle$ as the amount of angular momentum flown through $z=0$ during the time spent by the radiation to pass through the crystal. The fact that Eq. (58) contains n_o only (and not n_e) is related to the structure of the paraxial field traveling along the optical axis of the crystal and, more specifically, to its main plain wave (with wave vector $k_0 n_o \hat{\mathbf{e}}_z$) whose velocity is n_0/c.

VII. CONCLUSIONS

We have discussed the conservation of optical angular momentum in a uniaxial crystal, demonstrating that a balance equation can be derived only for the component of the radiation angular momentum along the optical axis, say the z axis; besides, we have pointed out that this situation is intimately related to the rotational invariance of the crystal around its unique optical axis. Consequently, we have furnished the proper expressions for the z component of angular momentum and the angular momentum flux of the radiation propagating in the crystal. We have specialized these general expressions to the case of a paraxial beam propagating along the optical axis and have demonstrated that the radiation angular momentum in the crystal has the same expression as the isotropic counterpart. In particular, the distinction between the intrinsic and the orbital parts of the angular momentum flux is the same as that existing for a beam propagating in a isotropic medium. Investigating the z evolution of these quantities, we have found that the dynamics of the intrinsic and orbital contributions reveals the existence of an exchange between them, a noticeable feature which is completely absent in the isotropic case. It is also interesting to note that this exchange saturates for $z\to\infty$ and we have predicted the asymptotical values of the two contributions: the intrinsic part asymptotically vanishes whereas the orbital part equates the total amount of the angular momentum flux. As an example, we have investigated the two angular momentum fluxes dynamics in the case of circularly polarized beams. For these beams, the intrinsic part of the angular momentum flux is simply proportional to the difference between the energies of the left-hand and the right-hand circularly polarized components. This fact simplifies the investigation of the evolution of the angular fluxes. We have also evaluated the amount of the beam's angular momentum stored in the crystal and we have demonstrated that it is given by the total angular momentum flux multiplied by the time an ordinary plane wave takes to pass through the crystal.

APPENDIX

1. Derivation of Eq. (27)

Equation (26) can be conveniently rewritten as

$$\langle F_{zz}\rangle = n_o\frac{\varepsilon_0 c}{2\omega}\mathrm{Re}\left[ix\left(A_x\frac{\partial A_y^*}{\partial x}-A_y\frac{\partial A_x^*}{\partial x}\right) + iy\left(A_x\frac{\partial A_y^*}{\partial y}-A_y\frac{\partial A_x^*}{\partial y}\right) + iA_x\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)A_x^* + iA_y\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)A_y^*\right], \quad (A1)$$

where use has been made of the relation $k_0/(2\omega^2\mu_0) = \varepsilon_0 c/(2\omega)$. In order to handle this expression we resort to the identity (easily deduced with some algebra)

$$\mathrm{Re}\left[ix\left(A_x\frac{\partial A_y^*}{\partial x}-A_y\frac{\partial A_x^*}{\partial x}\right)+iy\left(A_x\frac{\partial A_y^*}{\partial y}-A_y\frac{\partial A_x^*}{\partial y}\right)\right] = \frac{1}{i}(A_xA_y^* - A_x^*A_y) + \boldsymbol{\nabla}_\perp\cdot\mathbf{U}_\perp, \quad (A2)$$

where we introduced the transverse vector $\mathbf{U}_\perp = \mathrm{Im}(A_x^*A_y)(x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y)$. Combining Eqs. (A2) and (A1) we obtain

$$\langle F_{zz}\rangle = n_o \frac{\varepsilon_0 c}{2\omega}\left\{ \frac{1}{i}(A_x A_y^* - A_x^* A_y) + \mathrm{Re}\left[iA_x\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_x^* + iA_y\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_y^*\right] + \boldsymbol{\nabla}_\perp \cdot \mathbf{U}_\perp \right\}, \tag{A3}$$

which, inserted into Eq. (19), yields

$$\Phi(z) = n_o \frac{\varepsilon_0 c}{2\omega}\left[\int d^2\mathbf{r}_\perp\, i(A_x A_y^* - A_x^* A_y) - \mathrm{Re}\int d^2\mathbf{r}_\perp\, iA_x\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_x^* - \mathrm{Re}\int d^2\mathbf{r}_\perp\, iA_y\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)A_y^*\right]. \tag{A4}$$

Note that we have dropped the term $\boldsymbol{\nabla}_\perp \cdot \mathbf{U}_\perp$ of Eq. (A3), since its integral over the whole transverse plane vanishes because of the Green's theorem and the fact that the field vanishes at infinity. It is simple to show that the second and third integrals of Eq. (A4) are real quantities so that we can drop Re symbol. After integrating by parts these two integrals, Eq. (27) is obtained.

2. Derivation of Eqs. (34)

The paraxial field in Eq. (31) can be expressed as

$$\mathbf{A}_\perp(\mathbf{r}_\perp, z) = \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\tilde{\mathbf{a}}_\perp(\mathbf{k}_\perp, z), \tag{A5}$$

where

$$\tilde{\mathbf{a}}_\perp = [e^{-(iz/2k_0 n_o)k_\perp^2}\hat{P}_o + e^{-(in_o z/2k_0 n_e^2)k_\perp^2}\hat{P}_e]\tilde{\mathbf{A}}_\perp. \tag{A6}$$

Inserting Eq. (A5) into Eqs. (29) and exploiting some properties of the Fourier integral (generalized Parseval theorem) we obtain, after some algebra,

$$\Phi_I = i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp \tilde{\mathbf{a}}_\perp^\dagger \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tilde{\mathbf{a}}_\perp,$$

$$\Phi_O = i\gamma(2\pi)^2\int d^2\mathbf{k}_\perp\left[k_x\frac{\partial\tilde{\mathbf{a}}_\perp^\dagger}{\partial k_y}\tilde{\mathbf{a}}_\perp - k_y\frac{\partial\tilde{\mathbf{a}}_\perp^\dagger}{\partial k_x}\tilde{\mathbf{a}}_\perp\right]. \tag{A7}$$

Substituiting Eq. (A6) into the first of Eqs. (A7) and exploiting the properties $\hat{P}_o^2 = \hat{P}_o$, $\hat{P}_e^2 = \hat{P}_e$, and $\hat{P}_o\hat{P}_e = \hat{P}_e\hat{P}_o = 0$ of the projectors, the first of Eqs. (34) is easily obtained. In order to handle the second of Eqs. (A7), we use the relations

$$k_y\left(\frac{\partial\hat{P}_o}{\partial k_x}\hat{P}_o + \frac{\partial\hat{P}_e}{\partial k_x}\hat{P}_e\right) = \frac{1}{k_\perp^2}\begin{pmatrix} 0 & k_y^2 \\ -k_y^2 & 0\end{pmatrix},$$

$$k_x\left(\frac{\partial\hat{P}_o}{\partial k_y}\hat{P}_o + \frac{\partial\hat{P}_e}{\partial k_y}\hat{P}_e\right) = \frac{1}{k_\perp^2}\begin{pmatrix} 0 & -k_x^2 \\ k_x^2 & 0\end{pmatrix},$$

$$k_x\frac{\partial\hat{P}_e}{\partial k_y}\hat{P}_o - k_y\frac{\partial\hat{P}_e}{\partial k_x}\hat{P}_o = -\hat{Q}_o,$$

$$k_x\frac{\partial\hat{P}_o}{\partial k_y}\hat{P}_e - k_y\frac{\partial\hat{P}_o}{\partial k_x}\hat{P}_e = -\hat{Q}_e. \tag{A8}$$

Substituting Eq. (A6) into the second of Eqs. (A7) and taking Eqs. (A8) into account, the second of Eqs. (34) is obtained after some tedious but straightforward algebra.

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